Magic Method of Factoring a Quadratic Trinomial

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# Introduction

There are several methods on the web touted as “Magic Method” other than the one described in this paper. The one discussed in this paper has previously been referred to as the Bottoms-Up Method, Slide Method, Slip-and-Slide Method, and Slide-and-Divide Method.

# Preliminaries

*ax*2 + *bx* + *c*  is known as a quadratic form due to the *x*2 term. Since it is a polynomial with 3 terms, it is a trinomial. Hence it is known as the *quadratic trinomial*.

When *a* = 1, it is well known that *x*2 + *bx* + *c*  can be factored if there are factors of *c* that add up to *b.* Specifically, if *a*1 and *c*1 are such factors, then *ax*2 + *bx* + *c* = (*x* + *a*1) (*x* + *c*1).

# Factoring a Quadratic Trinomial by the Magic Method

We wish to factor *ax*2 + *bx* + *c*, where *a*, *b*, and *c* are relatively prime integers. There are at least 3 different sets of steps that all describe the same method. All start with the same first step. All involve reducing the pairs (*a*, *a*1) and (*a*, *c*1) as much as possible. Finally, all conclude with the same factoring formula.

**Magic Method 1.**

**Step 1**. Factor *x*2 + *bx* + a*c*

1. Find factors, *a*1 and *c*1, of *ac* that add up to *b*
2. Then *x*2 + *bx* + a*c*  = (*x* + *a*1) (*x* + *c*1)

**Step 2**. Divide the constants *a*1 and *c*1 by *a* and reduce as much as possible:



**Step 3**. Multiply by the respective denominators, *s* and *r* :

*ax*2 + *bx* + *c* = 

**Magic Method 2.**

**Step 1’**: Find factors, *a*1 and *c*1, of *ac* that add up to *b*, and write 

**Step 2’**: Reduce each factor as much as possible:

.

**Step 3’**: Divide by *a*. It turns out that *a* = *rs*, so we get

*ax*2 + *bx* + *c* = 

**Magic Method 3.**

**Step 1”**: Find factors, *a*1 and *c*1, of *ac* that add up to *b*, and write 

**Step 2”**: Divide each factor by as much as possible: .

**Step 3”**: Cancel:

*ax*2 + *bx* + *c* = 

*Note*. The Magic Method does not work if *a*, *b*, and *c* are not relatively prime. However, this is not a problem since the First Rule of Factoring is to factor out the Greatest Common Factor (GCF), leaving *a*, *b*, and *c* relatively prime.

# Examples of Magic Method 1

**Example 1**. Factor 12*x*2 – 17*x* + 6

Step 1. *ac* = (12) (6) = 72. Factors of 72 that add up to -17 are -8 and -9. So,

(*x* – 8) (*x* – 9)

Step 2. 

Step 3. 12*x*2 – 17*x* + 6 = (3*x* – 2) (4*x* – 3)

**Example 2**. Factor 2*x*2 + 7*x* + 6

Step 1. *ac* = (2) (6) = 12. Factors of 12 that add up to 7 are 3 and 4. So,

(*x* + 3) (*x* + 4)

Step 2. 

Step 3. 2*x*2 + 7*x* + 6 = (2*x* + 3) (*x* + 2)

# Proof of Magic Method:

We are given *ax*2 + *bx* + *c* where *a*, *b*, and *c* are relatively prime integers. We also *assume* that there are factors *a*1 and *c*1 of *ac* that add up to *b* because, otherwise, we know from the *ac* method that *ax*2 + *bx* + *c* is prime (i.e., it cannot be factored). Thus,

1. *a*1*c*1 = *ac*

and

(2) *a*1 + *c*1 = *b*.

Therefore

(1’) 

and

(2’)  .

That is,



Therefore, these are the factors of a quadratic trinomial with *a* = 1; namely

(3) 

Hence

(4) 

Define

(5) *r*  ≡ GCF(*a*, *a*1)

and

(6) *s* ≡ .

Since *r* is a factor of *a*, we have that *s* is an integer, and

(6’) *rs* = *a*.

Since *r* is also a factor of *a*1, define an integer

(7) *a*2 ≡ ,

which means that

(7’) *a*1 = *a*2*r*.

Also, define a rational number (i.e., not necessarily an integer)

(8) ,

which means that

(8’) *c*1 = *c*2*s*.

Substituting *rs* for *a* into (4) yields

,



.

which is the Magic Formula.

It might seem that the proof ends here. Indeed, we have shown how the “magic” steps lead to the correct answer. However, there remains 2 details that need proving.

**I.**  Factorization requires integers. We have shown that *s*, *a*2, and *r* are integers. We must show that *c*2 is an integer.

**II.** In Step 2 we were instructed to reduce the fraction  as much as possible to get . But neither *r* nor *c*2 were defined in terms of reducing the fraction as much as possible. We must show that  cannot be further reduced.

To better understand why we must prove detail II, it is instructive to check that equation (15) is correct by multiplying out the terms in parentheses on the right-hand side:

(16) 

 ✔

As we see, we used the fact that *rs* = *a* in (16). That is why, in Step 2, when we reduce  as much as possible we need to end up with *r* in the denominator. We must prove that we cannot reduce further.

We prove the 2 remaining details now.

**Detail I**:  *c*2 is an integer

By definition (8), this is equivalent to showing

(#) *s* is a factor of *c*1.

To prove (#), we first show

(##) *s* and *a*2 have no common factor other than 1.

For suppose that K is a common factor of *s* and *a*2. Then there are integers *A* and *B* such that

(9) *s* = *KA*

and

(10) *a*2 = *KB.*

Then

(11) *a* = *KrA*

because *,* and

(12) *a*1 = *KrB*

because . Thus, *Kr* is a common factor of *a* and *a*1. But, from (5), *r* is the greatest common factor of *a* and *a*1. Therefore, *K* = 1. Since *K* was allowed to be any common factor of *s* and *a*2, this proves (##), and hence

(13) GCF(*s, a*2) = 1.

Now, notice that

(14) .

This means that *s* is a factor of *a2c*1. Because of (##), that *s* and *a*2 have no common factor other than 1, all the factors of *s* must be factors of *c*1. That is, *s* is a factor of *c*1, which proves (#) and, hence, (Detail I).

**Detail II:** Reducing the fraction  as much as possible results in  .

“As much as possible” means we must reduce *c*1 and *a* by the GCF of *c*1 and *a*. Notice that we get from  to  by reducing numerator and denominator by *s:*

.

This shows that *s* is a factor of both *a* and *c*1. To finish the proof, we show that *s*  =  GCF(*a*, *c*1). Suppose some multiple of *s*, call it *sK*, where *K* is an integer, is a factor of both *a* and *c*1. We will show that *K* = 1 and that therefore *s*  = GCF(*a*, *c*1). Both

(17) 

and

(18) 

are integers since *sK* is a factor of both numerators. So, rewriting (17) yields

(\*) *a* = *K* (*As*).

Also,

(19) .

In addition,

(20) 

Thus,

(\*\*) .

Finally,

(\*\*\*) 

From (\*), (\*\*), and (\*\*\*), K is a factor of *a*, *b*, and *c*. Since *a*, *b*, and *c* are relatively prime, *K* = 1, which proves

(21) *s* = GCF(*a*, *c*1)

and concludes the proof.

# Summary

Ignoring Details I and II, the fundamentals of this proof are only 2 pages long using this very sparse formatting style. This part can be easily understood by a good high school student.

For the mathematically inclined, the key difficulty in this proof lies with *s* and how to incorporate the relative primeness of *a*, *b*, and *c* into the proof. Notice that  ** and also . The straight-forward approach would have been to **define** *s* as GCF(*a*, *c*1), in parallel with the definition of *r* in equation (5). In that case we would have had to prove that *rs* = *a*, and we would have had to invoke the relative primeness of *a*, *b*, and *c* to do this (not obvious how). Instead, the approach taken in this proof was to define *s* such that *rs* = *a* and prove **Detail I**, that *s* is a factor of both *a* and *c*1, and then use the relative primeness of *a*, *b*, and *c* to prove **Detail II**, that *s* = GCF(*a*, *c*1). This is easy to follow but was somewhat difficult to discover.